



Identifying weak focus and center in a convection model[☆]



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ABSTRACT

In this paper, we consider a 3-dimensional convection model and identify its weak focus and center. This model is described by quadratic polynomial differential equations, having three equilibria, two of them may be center-focus type. Center manifold theory and normal form theory are applied to prove that at least three limit cycles can bifurcate from a Hopf critical point around one of the two equilibria.

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1. Introduction

Identifying whether a singular point in a planar differential system is a center or focus has received great attention for more than one century and is still a hot topic. The order of the degeneracy of a singular point, related to Lyapunov constants in the polynomial ring of coefficients over the real field, determines the number of limit cycles bifurcating from the singular point. Poincaré [1] initiated the research of limit cycles, and later, the well-known Hilbert's 16th problem [2] promoted the development of this study. The second part of Hilbert's 16th problem is to study the upper bound on the number of limit cycles in planar polynomial systems of degree n , and the upper bound is called Hilbert number, denoted by $H(n)$. For general quadratic polynomial systems, the best result is four limit cycles with (3, 1) distribution, which was obtained 40 years ago [3], showing that $H(2) \geq 4$. For cubic polynomial systems, many results have been obtained on the lower bound of the Hilbert number [4,5]. So far, the best result obtained for cubic systems is $H(3) \geq 13$ [6,7]. Recently, many researchers paid attention to the center-focus problem for 3-dimensional (3-d) dynamical systems, for example, see [8,9]. In general, to study bifurcation in higher-dimensional

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systems, the system is first reduced to a simpler one on its center manifold (e.g., see [8,10]), and then to study the limit cycle bifurcation restricted to the center manifold. It is difficult to investigate Hopf bifurcation in general n -dimensional systems, while many results have been obtained for 3-d dynamical systems [11–13]. Very recently, Guo et al. found 12 limit cycles in a class of 3-d quadratic systems with Z_3 symmetry [14].

The convection model we will consider in this paper is described by the following ordinary differential equations [15]:

$$\begin{aligned}\dot{P} &= r_1 q_1 - \chi P - \eta_1 N - \eta_2 N^2 + \eta_3 P N - \varphi_1 F - \varphi_2 F^2 + \varphi_3 P F, \\ \dot{N} &= N(\gamma P - \beta_1 - \beta_2 N - \alpha_1 F), \\ \dot{F} &= F(\alpha_2 N - \mu),\end{aligned}\tag{1}$$

where P is the average potential energy related to the pressure profile, N is the average fluctuation energy and F is the zonal flow energy; $r_1 q_1$ and χ represent the source and diffusion, respectively. Simulation has shown that the fluctuation energy N can cause decrease in P , expressed by $-(\eta_1 N + \eta_2 N^2)$ taking linear and quadratic effects. However, as N initially begins to increase, P also increases, yielding a small jump, which is measured by $\eta_3 P N$. When the pressure gradient becomes sufficiently large, the equilibrium solution of the system (in S-shape) becomes unstable and a fluctuating flow is generated, which is described by $\gamma P N$. The two terms $-(\beta_1 N + \beta_2 N^2)$ describe the effect of the dissipation that causes the fluctuation energy N to be self-damped. α_1 and α_2 measure the rates of mutual effects of N and F , causing N to decrease and F to increase, respectively. The value of α_1 is usually estimated by a linear fit from plotting $\gamma P_H - \beta_1 - \beta_2 N_H$ as a function in F_H , and the value of α_2 is chosen to approximately reproduce the frequency of oscillations. μ is a debugging parameter, and the coefficients φ_1 , φ_2 and φ_3 are chosen to obtain the best approximation of F_H . Note that in system (1) the state variables P , N and F are nonnegative, and all the parameters take positive real values.

In [15], the authors carried out a bifurcation analysis on system (1) and studied the stability of equilibrium solutions and numerically demonstrated the existence of limit cycles. However, the number of bifurcating limit cycles and the center-focus conditions were not considered in [15]. It has been noted that identifying weak foci and centers in 3-d systems can cause great complexity in computation (e.g. see [16]). Moreover, the criteria for determining whether a singular point in planar dynamical systems is a center or focus, such as time-reversibility and integrability [17], are no longer applicable for 3-d systems, because application of center manifold theory to higher-dimensional dynamical systems merely yields approximation on invariant manifolds. To determine the number and stability of limit cycles bifurcating from a singular point, it needs to compute Lyapunov constants, which are usually obtained by applying one of the three methods: the method of normal forms [8,18,19], Poincaré return map [20], and Lyapunov function method [21,22]. For 3-d dynamical systems associated with Hopf bifurcation, we can combine center manifold theory and normal form theory to compute Lyapunov constants L_k with the aid of a computer algebra system such as Maple or Mathematica (e.g., see [8,19,23]).

2. Simplification of model (1)

To simplify the analysis for model (1), we first introduce the following scalings,

$$\begin{aligned}P &\rightarrow x_1, \quad N \rightarrow x_2, \quad F \rightarrow x_3, \quad r_1 \rightarrow r^2 \mu, \quad q_1 \rightarrow q^2, \quad \chi \rightarrow a_1^2 \mu, \quad \eta_1 \rightarrow n_1^2 \mu, \\ \eta_2 &\rightarrow n_2^2 \mu, \quad \eta_3 \rightarrow n_3^2 \mu, \quad \varphi_1 \rightarrow c_1^2 \mu, \quad \varphi_2 \rightarrow c_2^2 \mu, \quad \varphi_3 \rightarrow c_3^2 \mu, \quad \beta_1 \rightarrow b_1^2 \mu, \\ \beta_2 &\rightarrow b_2^2 \mu, \quad \gamma \rightarrow a_2^2 \mu, \quad \alpha_1 \rightarrow a_3^2 \mu, \quad \alpha_2 \rightarrow a_4^2 \mu, \quad t \rightarrow \frac{1}{\mu} t,\end{aligned}\tag{2}$$

under which system (1) becomes

$$\begin{aligned}\dot{x}_1 &= q^2 r^2 - a_1^2 x_1 - n_1^2 x_2 - c_1^2 x_3 - n_2^2 x_2^2 - c_2^2 x_3^2 + c_3^2 x_1 x_3 + n_3^2 x_1 x_2, \\ \dot{x}_2 &= x_2 (a_2^2 x_1 - b_1^2 - b_2^2 x_2 - a_3^2 x_3), \\ \dot{x}_3 &= x_3 (a_4^2 x_2 - 1),\end{aligned}\tag{3}$$

where the state variables $x_i, i = 1, 2, 3$ are nonnegative, and all the parameters $q, r, n_1, n_2, a_i (i = 1, 2, 3, 4), b_1, b_2$ and $c_i (i = 1, 2, 3)$ take nonzero real values. Note that we have transformed the original positive parameters to nonzero parameters for convenience in computation.

It is easy to show that system (3) has three equilibrium solutions:

$$E_s = \left(\frac{r^2 q^2}{a_1^2}, 0, 0 \right), \quad E_1 = \left(\frac{b_2^2 I_1 + b_1^2}{a_2^2}, I_1, 0 \right), \quad E_2 = \left(\frac{a_3^2 I_2 + b_1^2 a_4^2 + b_2^2}{a_2^2 a_4^2}, \frac{1}{a_4^2}, \frac{I_2}{a_4^2} \right),$$

where I_1 and I_2 are determined from the following quadratic polynomial equations:

$$\begin{aligned} (a_2^2 n_2^2 - b_2^2 n_3^2) I_1^2 + (a_1^2 b_2^2 + a_2^2 n_1^2 - b_1^2 n_3^2) I_1 - r^2 q^2 a_2^2 + a_1^2 b_1^2 &= 0, \\ (a_2^2 c_2^2 - a_3^2 c_3^2) I_2^2 + [(a_1^2 a_3^2 + a_2^2 c_1^2 - b_1^2 c_3^2) a_4^2 - a_3^2 n_3^2 - b_2^2 c_3^2] I_2 & \\ (a_1^2 b_1^2 - a_2^2 q^2 r^2) a_4^4 + (a_1^2 b_2^2 + a_2^2 n_1^2 - b_1^2 n_3^2) a_4^2 - b_2^2 n_3^2 + n_2^2 a_2^2 &= 0. \end{aligned} \tag{4}$$

The solutions of I_1 and I_2 obtained from the above two polynomial equations can be classified into two categories under the conditions: (A) $a_2^2 n_2^2 - b_2^2 n_3^2 = a_2^2 c_2^2 - a_3^2 c_3^2 = 0$; and (B) $a_2^2 n_2^2 \neq b_2^2 n_3^2$ or $a_2^2 c_2^2 \neq a_3^2 c_3^2$. For Case (A) the model can have three equilibria; while for Case (B), there may exist at most five equilibria. In this paper, we focus on Case (A) for which the three equilibria may be stable for certain parameter values. Therefore, solving the equation $a_2^2 n_2^2 - b_2^2 n_3^2 = 0$ for a_2 and $a_2^2 c_2^2 - a_3^2 c_3^2 = 0$ for c_2 we obtain

$$a_2 = \frac{b_2^2 n_3^2}{n_2^2}, \quad c_2 = \frac{a_3^2 c_3^2 n_2^2}{b_2^2 n_3^2},$$

which are substituted into system (3) to yield the following 3-d quadratic system,

$$\begin{aligned} \dot{x}_1 &= q^2 r^2 - a_1^2 x_1 - n_1^2 x_2 - c_1^2 x_3 - n_2^2 x_2^2 - \frac{a_3^2 c_3^2 n_2^2}{b_2^2 n_3^2} x_3^2 + n_3^2 x_1 x_2 + c_3^2 x_1 x_3, \\ \dot{x}_2 &= x_2 \left(\frac{b_2^2 n_3^2}{n_2^2} x_1 - b_1^2 - b_2^2 x_2 - a_3^2 x_3 \right), \\ \dot{x}_3 &= x_3 (a_4^2 x_2 - 1). \end{aligned} \tag{5}$$

Then, the equilibrium solutions of system (5) can be rewritten (with new names E_L and E_H) as

$$E_s = \left(\frac{r^2 q^2}{a_1^2}, 0, 0 \right), \quad E_L = \left(\frac{S_1 n_2^2}{S_3 b_2^2}, \frac{S_2}{S_3}, 0 \right), \quad E_H = \left(\frac{n_2^2 F_1}{F_2 a_4^2 b_2^2 n_3^2}, \frac{1}{a_4^2}, \frac{a_4^2 S_2 - S_3}{F_2} \right),$$

where

$$\begin{aligned} S_1 &= b_2^4 q^2 r^2 + b_1^2 (b_2^2 n_1^2 - b_1^2 n_2^2), \\ S_2 &= b_2^2 n_3^2 q^2 r^2 - a_1^2 b_1^2 n_2^2, \\ S_3 &= a_1^2 b_2^2 n_2^2 + n_3^2 (b_2^2 n_1^2 - b_1^2 n_2^2), \\ F_1 &= (a_4^2 c_1^2 n_3^2 - c_3^2 n_2^2) b_2^4 + [(a_3^2 q^2 r^2 + b_1^2 c_1^2) n_3^2 a_4^4 - (a_3^2 n_1^2 n_3^2 + 2b_1^2 c_3^2 n_2^2) a_4^2 - a_3^2 n_2^2 n_3^2] b_2^2 - a_4^4 b_1^4 c_3^2 n_2^2, \\ F_2 &= [(a_1^2 a_3^2 - b_1^2 c_3^2) a_4^2 - a_3^2 n_3^2 - b_2^2 c_3^2] n_2^2 + a_4^2 b_2^2 c_1^2 n_3^2. \end{aligned}$$

It is easy to see that the equilibrium E_s exists for any positive parameter values; the equilibrium E_L exists if $S_1/S_3 > 0$ and $S_2/S_3 > 0$; and the equilibrium E_H exists if $F_1/F_2 > 0$ and $(a_4^2 S_2 - S_3)/F_2 > 0$. Further, note that both E_L and E_H are possible to be of center-focus type, while E_s can be only node or saddle. We apply the method of normal forms to prove that system (5) can have three small-amplitude limit cycles bifurcating from E_H . Moreover, it can be shown that the equilibrium E_L is a center, restricted to a global center manifold,

3. Weak focus at the equilibrium E_H

In [15], the authors present some qualitative properties for the equilibria of system (1). The reduced system (5) with less number of parameters helps us in simplifying the computation of normal forms and focus values. In the following, we first consider the qualitative properties of E_s . The Jacobian matrix of system (5) evaluated at E_s is given by

$$J(E_s) = \begin{bmatrix} -a_1^2 & \frac{n_3^2 q^2 r^2 - a_1^2 n_1^2}{a_1^2} & c_3^2 q^2 r^2 - a_1^2 c_1^2 a_1^2 \\ 0 & \frac{S_2}{a_1^2 n_2^2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (6)$$

which has eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = -a_1^2, \quad \lambda_3 = \frac{S_2}{a_1^2 n_2^2}.$$

All the three eigenvalues are real, and λ_1 and λ_2 are negative constants. Thus, E_s is a stable node if $S_2 < 0$ and a saddle if $S_2 > 0$.

Next, consider the equilibrium E_H . In [15], the stability of equilibria was discussed, and numerical bifurcation software was used to plot bifurcation diagrams where Hopf bifurcation was identified, which is based on a linear analysis. In this section, we will further investigate the Hopf bifurcation in system (5), and prove the existence of three limit cycles. In order for system (5) to have a Hopf critical point on the equilibrium solution E_H and make the computation of focus values manageable, we further let

$$\begin{aligned} r = c_1 = a_3 = a_1 = 1, \quad q = \frac{\sqrt{2}}{c_3}, \quad b_1 = \frac{b_2 n_3}{c_3 n_2}, \\ S_4 = a_4^2 b_2^2 n_3^2 - b_2^2 c_3^2 n_2^2 + a_4^2 n_2^2 - n_2^2 n_3^2, \\ n_1 = \frac{\sqrt{S_4 [a_4^2 b_2^2 n_3^4 (a_4^2 + b_2^2 - n_3^2) + (2n_3^4 - b_2^2 n_3^2 - c_3^2 n_2^2) S_4]}}{S_4 n_3 c_3}. \end{aligned} \quad (7)$$

Thus, under the above conditions, the linearized system of (5) evaluated at the E_H contains a non-zero real eigenvalue and two purely imaginary eigenvalues. Next, introducing the transformation,

$$\begin{aligned} x_1 &= \frac{a_4^2 n_2^2 n_3^2 (a_4^2 + b_2^2 - n_3^2) + (a_4^2 n_3^2 + c_3^2 n_2^2) S_4}{S_4 a_4^2 c_3^2 n_3^2} + \frac{n_2^2}{n_3^2} z_1 + \frac{a_4^2 b_2 n_2 (a_4^2 n_3^2 - c_3^2 n_2^2 - n_2^2)}{S_4 n_3 c_3} z_2 \\ &\quad + \frac{n_2^2 F_3}{S_4 a_4^4 b_2^4 n_3^4} z_3, \\ x_2 &= \frac{1}{a_4^2} + z_1 - \frac{n_2^2 c_3^2 (a_4^2 - b_2^2 c_3^2 - n_3^2)}{a_4^4 b_2^2 n_3^2} z_3, \\ x_3 &= \frac{b_2^2 n_3^2 (a_4^2 + b_2^2 - n_3^2)}{S_4 c_3^2} - \frac{b_2 a_4^2 n_2 n_3 (a_4^2 + b_2^2 - n_3^2)}{S_4 c_3} z_2 + z_3, \\ t &\rightarrow \frac{c_3 n_2}{b_2 n_3} t, \end{aligned} \quad (8)$$

where

$$\begin{aligned} F_3 &= \{ a_4^4 b_2^2 n_3^2 - b_2^2 n_2^2 (a_4 - n_3)(a_4 + n_3) c_3^4 \\ &\quad + [(-b_2^2 n_3^2 + n_2^2) a_4^4 - n_3^2 (b_2^4 - b_2^2 n_3^2 + 2n_2^2) a_4^2 + n_2^2 n_3^4] c_3^2 \} S_4 + a_4^4 b_2^4 c_3^2 n_3^4 (a_4^2 + b_2^2 - n_3^2), \end{aligned}$$

into (5), we obtain

$$\begin{aligned} \dot{z}_1 &= z_2 + A_{002}z_3^2 + A_{011}z_2z_3 + A_{020}z_2^2 + A_{101}z_1z_3 + A_{110}z_1z_2, \\ \dot{z}_2 &= -z_1 + B_{002}z_3^2 + B_{011}z_2z_3 + B_{020}z_2^2 + B_{101}z_1z_3 + B_{110}z_1z_2, \\ \dot{z}_3 &= C_{001}z_3 + C_{002}z_3^2 + C_{011}z_2z_3 + C_{020}z_2^2 + C_{101}z_1z_3 + C_{110}z_1z_2, \end{aligned} \tag{9}$$

where the coefficients A_{ijk} , B_{ijk} and C_{ijk} are given in the website: <http://www.apmaths.uwo.ca/~pyu>.

Clearly, under the conditions given in (7), the singular point E_H of system (5) corresponds to the origin of (9), which is a Hopf-type critical point. We have the following result.

Theorem 3.1. *Under the conditions given in (7), there exist parameter values for system (1) such that at least three limit cycles can bifurcate from E_H within some, hence every, two-dimensional invariant manifold through E_H .*

Proof. In order to make the computation of focus values manageable and find the parameter values satisfying the conditions, we further set $a_4 = 10$ and $n_2 = n_3 = 1$ in system (9). Then, we apply center manifold theory and the method of normal forms, as well as the Maple program in [23] to system (9) to obtain the focus values. In particular, v_1 and v_2 are given by

$$\begin{aligned} v_1 &= -\frac{125000(b_2^2c_3^2 - 100b_2^2 - 99)b_2(b_2^2 + 99)c_3^3F_4}{(b_2^2c_3^2 - 99)F_6F_7}, \\ v_2 &= -\frac{312500000(b_2^2c_3^2 - 100b_2^2 - 99)(b_2^2 + 99)c_3^3F_5}{9b_2(b_2^2c_3^2 - 99)^3F_6^3F_7^3F_8}, \end{aligned}$$

where the lengthy expressions of the polynomials $F_i, i = 4, 5, \dots, 8$ are given in terms of b_2 and c_3 , and can be found in the website: <http://www.apmaths.uwo.ca/~pyu>.

In order to obtain maximal number of small-amplitude limit cycles bifurcating from the origin of system (9), we use the coefficients b_2 and c_3 to solve the two polynomial equations, $F_4 = F_5 = 0$. If a solution of $F_4 = F_5 = 0$ yields $v_3 \neq 0$, we may obtain two small-amplitude limit cycles by properly perturbing b_2 and c_3 . To achieve this, we first compute the resultant of F_4 and F_5 , yielding

$$\begin{aligned} F_{45} &= 4.45990583806 \cdots \times 10^{940} c_3^{124} (c_3^2 - 99)^{56} (c_3^4 - 200c_3^2 + 299)^4 \\ &\quad \times (275c_3^{10} - 109175c_3^8 - 2128419c_3^6 + 3004576243c_3^4 - 137557848532c_3^2 + 71109368756)^4 \\ &\quad \times (33275c_3^{14} - 13105026c_3^{12} + 464987348c_3^{10} + 6341100342c_3^8 - 7908729138339c_3^6 \\ &\quad + 296555246006244c_3^4 - 265954686172548c_3^2 + 982858205776)^4 F_{45a}^2, \end{aligned}$$

where F_{45a} is a polynomial in c_3^2 , given in the website: <http://www.apmaths.uwo.ca/~pyu>.

Now, to find the solutions of $F_4 = F_5 = 0$, we must solve $F_{45} = 0$. It is seen from F_{45} that besides the factor F_{45a} , there are five polynomial factors which may also yield solutions c_3^2 for $F_{45} = 0$. However, a careful verification shows that the solutions solved from these five factors do not satisfy the equations $F_4 = F_5 = 0$. Thus, possible solutions for $F_{45} = 0$ only come from the positive roots of F_{45a} (since $c_3^2 > 0$). It can be shown that the polynomial F_{45a} has 7 positive roots for c_3^2 , which in turn yield corresponding 14 solutions for b_2 , among them only eight sets of solutions c_3 and b_2 satisfy the equations, $F_4 = F_5 = 0$. Then, under the conditions, $C_{001} < 0$ and $N_1 \geq 0$, we choose one of the solutions as follows:

$$b_2 = -0.53868839285 \cdots, \quad c_3 = -0.61448088908 \cdots,$$

under which

$$v_1 = v_2 = 0, \quad v_3 = -5.92361180765 \cdots \times 10^8 \neq 0.$$

Moreover, a direct calculation shows that the Jacobian evaluated at the critical values is

$$\det \begin{bmatrix} \frac{\partial(v_1, v_2)}{\partial(b_2, c_3)} \end{bmatrix} = -6.44212213245 \cdots \times 10^8 \neq 0,$$

implying that system (9) can indeed have two small-amplitude limit cycles bifurcating from the center-type singular point (the origin). Further, a linear perturbation on n_1 generates a third small-amplitude limit cycle. Thus, system (1) can have at least three limit cycles around the equilibrium E_H .

The proof of Theorem 3.1 is complete. \square

Finally, we point out that for the singular point E_L , one can identify the Hopf bifurcation condition and then apply normal form theory and global center manifold theory [24] to prove that when E_L is a Hopf singular point, it must be a center on a global center manifold. The proof is straightforward and is thus omitted.

4. Conclusion

In this paper, we have studied a convection model and paid particular attention on bifurcation of limit cycles. We have shown that the model has three distinct equilibria, and that there exist parameter values for the model to have three small-amplitude limit cycles around one of the equilibria.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.aml.2019.106019>.

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